The derivations here provide some background regarding the different strategies considered by the website. Four strategies are examined: Straight Line, Constant Heading, Dynamic Heading and the Dog Paddle.

Most of what appears here is basic algebra and geometry, although the Dog Paddle scenario involves some simple differential equations, and calculus appears occasionally. Even if your math is rusty, the derivations are very detailed, and you should get the gist. From a practical point of view, it's the conclusions that matter.

1 Straight Line

In this case, the boat travels in a straight line to the destination, steering so as to compensate for the current. This is the easiest strategy to understand, and it sets the stage for the additional strategies.

1.1 Constant Current

To begin, assume that the current is constant throughout the journey. The boat starts at $(0, 0)$ and the destination is at (w, d). It is assumed that $w > 0$ so that the destination is to the east of the starting point. Let v be the current and b be the speed of the boat relative to the water. The current is assumed to flow strictly in a north/south direction; there is no current flowing east/west. Let (α, β) be the boat's thrust vector; that is,

$$\begin{aligned} \alpha &= b\cos\theta \\ \beta &= b\sin\theta, \end{aligned}$$

where θ is the heading (with $\theta = 0$ corresponding to due east). Note that $b^2 = \alpha^2 + \beta^2$.

Let t be the time taken to reach the destination. Then

$$\begin{aligned} w &= \alpha t \\ d &= (\beta + v)t. \end{aligned}$$

From these two equations it is possible to derive values for α, β and t:

$$t = w/\alpha,$$

which implies

$$d\alpha = w(\beta + v).$$

Square both sides and make use of $\alpha^2 = b^2 - \beta^2$:

$$\begin{aligned} d^2\alpha^2 &= w^2(\beta + v)^2 \\ d^2(b^2 - \beta^2) &= w^2(\beta + v)^2. \end{aligned}$$

Expand this out, collect terms and apply the quadratic formula to obtain [1]

$$\beta = \frac{-vw^2 \pm \sqrt{d^2(w^2b^2 - v^2w^2 + d^2b^2)}}{w^2 + d^2}. \tag{1}$$

So, given the various inputs (v, w, *etc.*), this formula provides β, which in turn gives α and t. In a similar way,

$$\alpha = \frac{vwd \pm w\sqrt{w^2b^2 - v^2w^2 + d^2b^2}}{w^2 + d^2}. \tag{2}$$

If your algebra is rusty, then one of the things about Equation (1) that may be confusing is the $\pm$ in front of the square root. One way to to think about the two solutions given by Equation (1) is that the equation doesn't say anything about whether d is upstream or downstream; one of the two "solutions" will go a distance of $|d|$ upstream, and the other will go $|d|$ downstream.[2] Only one of these is the correct answer. In fact, things are a little more complicated than that.

The quantity under the square root (the "discriminant") must be positive since you can't take the square root of a negative number, and the discriminant is negative when v is large. In particular, if $|v| > (b/w)\sqrt{w^2 + d^2}$, then the discriminant is negative. Leaving the math aside, if the boat can go faster than the current, then the boat can go "anywhere," but if the boat is slower than the current, then there are limits on the points it can reach. It's useful to know what these limits are.

By assumption,

$$d = (\beta + v)t,$$

and this can be rewritten as

$$d = \frac{w(\beta + v)}{\sqrt{b^2 - \beta^2}}.$$

Let $f(\beta) = d$ be this function for d in terms of β. The values for $f(\beta)$ will range over all possible points upstream and downstream that the boat can reach. We want to know what the extreme values are, so solve $f'(\beta) = 0$.

$$f'(\beta) = \frac{w(b^2 + \beta v)}{(b^2 - \beta^2)^{3/2}},$$

[1] At this point, it may be tempting to observe that β must be less than b, and see where that leads. The short answer is, "nowhere." What it leads to is

$$(d^2 + w^2)(v + b)^2 > 0,$$

which is trivially true.

[2] In fact, that's not the whole story. Suppose that the destination is downstream. The boat might go there quickly by pointing only slightly into the current, *or* it could fight the current more nearly head-on, and slowly inch across the current.

and this is zero iff

$$\beta = -b^2/v.$$

If $b > |v|$, then this extremum would occur when $\beta > b$, which is not possible (the boat can't go that fast). So the question of an extremum only makes sense when $|v| < b$. But we knew that anyway, since $b > |v|$ implies that the boat can go "anywhere."

Set $\beta_0 = -b^2/v$. Then

$$f(\beta_0) = \operatorname{sgn}(v)\,\frac{w}{b}\sqrt{v^2 - b^2}, \tag{3}$$

which provides a tidy expression for how the boat's destination is limited when $b < |v|$.

Note that if $b = v$ exactly, then we need to be a little more careful, for then $\beta_0 = -b$, and the boat makes no headway at all in the x-direction. In this case, we can get as close as we want to $f(\beta_0)$ (by making β closer and closer to b, so that the crossing takes longer and longer), but never quite reach it.

Finally, the expression for β includes terms in d^2 but not in d, and this can be confusing. Just because d is reachable, does not mean that $-d$ is reachable, even though the same value for β "solves" the equation in the two cases.

1.2 Variable Current

Assume that the current is divided into n vertical strips, where the current is constant in each strip. The total width is w, and the strips have widths $w_1 \dots w_n$, so that $w = \sum w_i$. The thrust varies from strip to strip. Let the thrust in strip i be given by (α_i, β_i), and let v_i be the current in each strip. Use t_i to represent the time taken to cross each strip. Then

$$\begin{aligned} w_i &= \alpha_i t_i \\ d_i &= (\beta_i + v_i)t_i, \end{aligned}$$

where d_i is the amount by which the boat's y-position changes as it crosses each strip. To reach the destination at (w, d), we must have $d = \sum d_i$.

Determining the (α_i, β_i) is very much as above, in the constant current case. The boat must follow the line from $(0, 0)$ to (w, d) and the thrust in each strip is chosen to compensate for the current, v_i. The line is given by

$$y = \frac{d}{w}\,x,$$

so that

$$d_i = \frac{d}{w}\,w_i,$$

and

$$\begin{aligned}
d_i &= (\beta_i + v_i)t_i \\
\frac{d}{w}\, w_i &= (\beta_i + v_i)t_i \\
\frac{d}{w}\, w_i &= (\beta_i + v_i)\,\frac{w_i}{\alpha_i} \\
d^2(b^2 - \beta_i^2) &= w^2(\beta_i + v_i)^2.
\end{aligned}$$

As before, this can be solved to obtain

$$\beta_i = \frac{-v_i w_i^2 \pm \sqrt{d_i^2(w_i^2 b^2 - v_i^2 w_i^2 + d_i^2 b^2)}}{w_i^2 + d_i^2}, \tag{4}$$

which has exactly the same from as in the constant current case.

1.3 Variable Current and Variable Boat Speed

Suppose that the speed of the boat is allowed to vary from strip to strip, with b_i equal to the boat's speed in strip i. The same algebra used to arrive at Equation (4) gives

$$\beta_i = \frac{-v_i w_i^2 \pm \sqrt{d_i^2(w_i^2 b_i^2 - v_i^2 w_i^2 + d_i^2 b_i^2)}}{w_i^2 + d_i^2}. \tag{5}$$

2 Constant Heading

If the current is constant throughout the width, then a boat which takes a constant heading will follow a straight line, and the situation is identical to what was considered in the previous section. So allow the current to vary over the width.

Set things up just as in the Straight Line case. There are n vertical strips with total width w. Each strip has width w_i, and current v_i. The time taken to cross each strip is t_i.

This case differs from the Straight Line strategy in that the thrust vector is constant throughout all the strips. The thrust is always (α, β), with no subscripts, so that

$$\begin{aligned}
w_i &= \alpha t_i \\
d_i &= (\beta + v_i)t_i,
\end{aligned}$$

where d_i is the amount by which the boat's y-position changes as it crosses each strip. To reach the destination at (w, d), we must have $d = \sum d_i$. So

$$\begin{aligned}
w &= \alpha \sum t_i \\
d &= \sum (\beta + v_i)t_i
\end{aligned}$$

The goal is to choose α and β to solve these equations, so that the boat arrives at the destination.

The algebra is very much as in the Straight Line case:

$$\begin{aligned} t_i &= w_i/\alpha \\ d &= \sum(\beta + v_i)(w_i/\alpha), \end{aligned}$$

so that

$$\begin{aligned} d\alpha &= \sum(\beta + v_i)w_i \\ &= \beta w + \sum v_i w_i. \end{aligned}$$

Set $V = \sum v_i w_i$. The same process that was used in the Straight Line case works here. Square both sides, collect like terms, and apply the quadratic formula to obtain

$$\beta = \frac{-wV \pm \sqrt{d^2(w^2b^2 - V^2 + d^2b^2)}}{w^2 + d^2}$$

2.1 Variable Boat Speed

Suppose that boat's speed is allowed to vary, so that b_i is the boat's speed in strip i. The algebra can be made a bit simpler by observing that

$$\begin{aligned} \alpha_i &= b_i \cos\theta \\ \beta_i &= b_i \sin\theta, \end{aligned}$$

for some heading, θ. The heading is constant for this strategy, so that θ is constant (*i.e.*, no subscripts are needed on θ). Set $\alpha = \cos\theta$ and $\beta = \sin\theta$.

Now

$$\begin{aligned} d &= \sum \frac{w_i}{\alpha_i}(\beta_i + v_i) \\ &= \sum \frac{w_i}{b_i\alpha}(b_i\beta + v_i), \end{aligned}$$

so that

$$\begin{aligned} d\alpha &= \sum \frac{w_i}{b_i}(b_i\beta + v_i) \\ &= \beta w + \sum \frac{w_i v_i}{b_i}. \end{aligned}$$

Set $\overline{V} = \sum v_i w_i / b_i$. The same reasoning as above indicates that

$$\beta = \frac{-w\overline{V} \pm \sqrt{d^2\left(w^2 - \overline{V}^2 + d^2\right)}}{w^2 + d^2},$$

and β_i is obtained from $\beta_i = b_i\beta$.

3 Dynamic Heading

In this case, allow the boat's heading to vary arbitrarily as it proceeds across the current. The goal is to choose the heading so as to minimize the total time required to cross. Such a strategy is "time-optimal." In the two previous cases (Straight Line and Constant Heading), there was no optimization involved; the constraints on how the course was chosen determine the total time required.

If the current is constant over the journey, then the optimal heading is the same as the straight line or constant heading strategies, so allow the current to vary in strips, using the same notation as in the two previous cases.

Unfortunately, I could find no compact algebraic expression for this strategy. Maybe I'm not smart enough, not patient enough, or don't know enough, but a naive first-year calculus approach won't work because the equations are too messy; and a more sophisticated calculus of variations approach doesn't work because the current velocity is assumed to follow a step function rather than being smooth. It might be possible to use some kind of a limit argument to go from smooth functions to step functions with the partial differential equation provided by the calculus of variations approach, but I lost patience.

In any case, Equations (4) and (5) are still valid, and this provides a means to find a solution numerically. For any given path, which moves from one strip to another at the "crossing points," Equation (4) or (5) allows the total time required to be calculated. These crossing points are given by the choice of d_i. Something like Brent's method can be used to vary these points to find the minimum total time. Assuming the inputs are reasonable (*i.e.*, no physically implausible current setups), this should lead to a global minimum rather than a local minimum.

3.1 Calm Edge Case

Even if there are only two strips, finding a nice expression for the optimal crossing point is impossible. As far as I can see, such an expression requires a solution to an eighth-degree polynomial. However, a situation intermediate between two strips and a single strip with constant current does have a reasonable solution.

Under this "calm edge" case, the boat crosses a strip of constant current, and, once it reaches the far side, it can travel up and down at full speed, without any effect from current. If the boat's thrust vector is given by (α, β), and the width of the current is w, then

$$t_1 = \frac{w}{\alpha}$$

is the time taken to cross the current. Let v be the velocity of the current. Then, the boat emerges on the far side with y-coordinate

$$\begin{aligned} d_1 &= t_1(\beta + v) \\ &= \frac{w}{\alpha}(\beta + v). \end{aligned}$$

The time required to travel along the calm edge to the destination at d is then

$$\begin{aligned} t_2 &= \frac{1}{b}|d - d_1| \\ &= \frac{1}{b}\left|d - \frac{w}{\alpha}(\beta + v)\right|, \end{aligned}$$

and the total time taken to reach the destination is $T = t_1 + t_2$. The goal is to minimize T.

The absolute value in the expression for t_2 makes this scenario tricky. There are exactly three situations leading to T being at a minimum: the boat heads directly for d so that $t_2 = 0$; or $t_2 \neq 0$ and the quantity in the absolute value is either positive or negative (d_1 is north or south of d). The most straightforward way to determine the minimum for T is to consider three distinct functions:

$$\begin{aligned} T^{(0)} &= \frac{w(w^2 + d^2)}{vwd \pm w\sqrt{w^2b^2 - v^2w2 + d^2b^2}} \\ T^{(+)} &= \frac{w}{\alpha} + \frac{1}{b}\left(d - \frac{w}{\alpha}(\beta + v)\right) \\ T^{(-)} &= \frac{w}{\alpha} - \frac{1}{b}\left(d - \frac{w}{\alpha}(\beta + v)\right), \end{aligned}$$

where $T^{(0)}$ is a constant equal to the time required to follow a direct course to d, as given by Equation (2). Find the minimums for $T^{(+)}$ and $T^{(-)}$, along with the value for $T^{(0)}$. Whichever of these is the least is the minimum possible value for T.

Set $S = \pm 1$, depending on whether $T^{(+)}$ or $T^{(-)}$ is being considered. If $T^{(S)}$ has a minimum, then it will occur at an α for which $dT^{(S)}/d\alpha$ is zero:

$$\begin{aligned} \frac{dT^{(S)}}{d\alpha} &= -\frac{w}{\alpha^2} + \frac{S}{b}\left(\frac{w}{\alpha^2}(\beta + v) - \frac{w}{\alpha}\frac{d\beta}{d\alpha}\right) \\ &= -\frac{w}{\alpha^2} + \frac{S}{b}\left(\frac{w}{\alpha^2}(\beta + v) + \frac{w}{\beta}\right). \end{aligned}$$

Set this to zero and solve for α, bearing in mind that β depends on α:

$$\begin{aligned} 0 &= -\frac{1}{\alpha^2} + \frac{S}{b}\left(\frac{1}{\alpha^2}(\beta + v) + \frac{1}{\beta}\right) \\ &= -b\beta + S\left(\beta(\beta + v) + \alpha^2\right) \\ &= -b\beta + S\left(\beta^2 + v\beta + \alpha^2\right) \\ &= -b\beta + S\left(b^2 + v\beta\right) \\ &= Sb^2 + \beta\left(Sv - b\right). \end{aligned}$$

Then,

$$\beta = \frac{Sb^2}{b - Sv}, \tag{6}$$

and

$$\begin{aligned}\alpha^2 &= b^2 - \beta^2 \\ &= b^2\left(1 - \frac{b^2}{(b-Sv)^2}\right) \\ &= \frac{b^2 v(v-2Sb)}{(b-Sv)^2}.\end{aligned}$$

Since α is always positive, this can be written as

$$\alpha = b\sqrt{\frac{v(v-2Sb)}{(b-Sv)^2}}.$$

When $S = +1$, these values for α and β indicate the extreme value for $T^{(+)}$; and when $S = -1$, they indicate the extreme value for $T^{(-)}$, assuming that such values exist, and bearing in mind that these extreme values may be maxima, not minima.

This is surprising. Nowhere in the expressions for α or β does d or w appear. Once d is so far away that a direct course to d is no longer the fastest, the best approach is entirely independent of just how far away d may be. The best heading depends only on the relative sizes of b and v.

4 The Dog Paddle

Consider a dog at (x, y) traveling toward a stick at $S = (0, 0)$ through a current flowing in the y-direction at rate v. Let the dog start the trip (time $t = 0$) at position (w, d). In earlier cases, the destination was at (w, d) and the dog started at $(0, 0)$, but the algebra will be simpler if the destination is at $(0, 0)$.

Let $D = (x(t), y(t))$ be the position of the dog. The dog always paddles directly toward the stick, although its course is modified by the current. Let (α, β) be the direction in which the dog paddles (the thrust vector), and require that $b^2 = \alpha^2 + \beta^2$, for some fixed b. The value, b, is the speed at which the dog swims in calm water. Taking the current into account,

$$\begin{aligned}D'(t) &= \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \\ &= (\alpha, \beta + v).\end{aligned}$$

Although the dog is swept along by the current, he always paddles toward the stick; that is, the thrust vector always points to the stick:

$$\frac{(\alpha, \beta)}{b} = \frac{S - D}{||S - D||} = \frac{-D}{||D||} = \frac{-(x, y)}{\sqrt{x^2 + y^2}}.$$

This provides explicit expressions for α and β in terms of x and y, hence an explicit expression for $D'(t)$.

The foregoing provides an expression relating $x(t)$ and $y(t)$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\
&= \frac{\beta + v}{\alpha} \\
&= \frac{\frac{-by}{\sqrt{x^2+y^2}} + v}{\frac{-bx}{\sqrt{x^2+y^2}}} \\
&= \frac{by - v\sqrt{x^2+y^2}}{bx}
\end{aligned}$$

Solving this differential equation involves a few steps. Set $u = y/x$ and make the substitutions

$$\begin{aligned}
y &= xu \\
\frac{dy}{dx} &= u + x\frac{du}{dx} \\
x^2 + y^2 &= x^2(1+u^2)
\end{aligned}$$

to obtain

$$\frac{dy}{dx} = \frac{bxu - v\sqrt{x^2(1+u^2)}}{bx}.$$

Assume that $x < 0$. This is reasonable; it means that the dog starts out to the left of the destination, and stays to the left. Then

$$\begin{aligned}
u + x\frac{du}{dx} &= \frac{bxu + vx\sqrt{1+u^2}}{bx} \\
u + x\frac{du}{dx} &= u + (v/b)\sqrt{1+u^2} \\
x\frac{du}{dx} &= (v/b)\sqrt{1+u^2}
\end{aligned}$$

The variables can now be separated:

$$\frac{b}{\sqrt{1+u^2}}\,du = \frac{v}{x}\,dx.$$

Integrating both sides yields

$$b\ln|u + \sqrt{1+u^2}| = v\ln|x| + C,$$

or, equivalently,

$$\begin{aligned}
\sinh^{-1} u &= (v/b)\ln|x| + C \\
&= \ln|Cx|^{v/b}, \qquad (7)
\end{aligned}$$

where C is a generic stand-in for the constant of integration. Recall that

$$\sinh z = \frac{e^z - e^{-z}}{2},$$

so that

$$\sinh(\ln z) = \frac{1}{2}\left(z - \frac{1}{z}\right),$$

and we have

$$u = \frac{1}{2}\left(|Cx|^{v/b} - |Cx|^{-v/b}\right).$$

Recalling that $u = y/x$, we now have

$$y = \frac{x}{2}\left(|Cx|^{v/b} - |Cx|^{-v/b}\right). \tag{8}$$

To determine C, go back to Equation (7) and recall that the dog starts at position $(x, y) = (w, d)$, where $w < 0$. So $u = d/w$ and

$$\begin{aligned}
\ln|u + \sqrt{1+u^2}| &= \ln|Cx|^{v/b} \\
|(d/w) + \sqrt{1+(d/w)^2}| &= |Cw|^{v/b} \\
|(d/w) + \sqrt{1+(d/w)^2}|^{b/v} &= |Cw| \\
\frac{1}{w}\left((d/w) + \sqrt{1+(d/w)^2}\right)^{b/v} &= C,
\end{aligned}$$

where dropping the absolute values on C is OK because the absolute value is applied in Equation (8).

There is one fly in the ointment. What happens when the dog reaches the destination, which is taken to be the origin? In this case, the expression for y given by Equation (8) appears to involve a division by zero. In fact, the x-term out front means that you are not really dividing by zero, but some care is needed when implementing this in a computer program.

It would also be useful to know the time at which various x-positions are reached. This requires the solution to a messy integral:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{-bx}{\sqrt{x^2+y^2}} \\
dt &= -\frac{\sqrt{x^2+y^2}}{bx}\ dx
\end{aligned}$$

so that

$$\begin{aligned}
t(x) &= -\int \frac{\sqrt{x^2+y^2}}{bx}\ dx \\
&= \frac{1}{b}\int \sqrt{1+(y/x)^2}\ dx, .
\end{aligned}$$

where the minus disappears in the second step because $x < 0$. According to Equation (8), this can be written as

$$t(x) = \frac{1}{b} \int \sqrt{1 + \left(\frac{1}{2} \left(|Cx|^{v/b} - |Cx|^{-v/b} \right) \right)^2} \, dx.$$

Set $k = v/b$, and drop the absolute value; if we assume that x does not change sign over the integration, it's OK to drop it (*i.e.*, $x < 0$ and take $C < 0$). Set $w = Cx$ so that $dw = Cdx$. Then

$$\begin{aligned}
t(x) &= \frac{1}{bC} \int \sqrt{1 + \left(\frac{1}{2} \left(w^k - \frac{1}{w^k} \right) \right)^2} \, dw \\
&= \frac{1}{bC} \int \sqrt{1 + \frac{1}{4} \left(w^{2k} - 2 + \frac{1}{w^{2k}} \right)} \, dw \\
&= \frac{1}{bC} \int \sqrt{\frac{1}{4} \left(w^{2k} + 2 + \frac{1}{w^{2k}} \right)} \, dw \\
&= \frac{1}{2bC} \int \sqrt{\left(w^k + \frac{1}{w^k} \right)^2} \, dw \\
&= \frac{1}{2bC} \int \left(w^k + \frac{1}{w^k} \right) \, dw
\end{aligned}$$

This is an integral we can do:

$$t(x) = \frac{1}{2bC} \left(\frac{w^{k+1}}{k+1} + \frac{w^{1-k}}{1-k} \right) + K, \tag{9}$$

for some constant of integration, K. Unwinding the substitutions gives

$$\begin{aligned}
t(x) &= \frac{1}{2bC} \left(\frac{(Cx)^{(v/b)+1}}{(v/b)+1} + \frac{(Cx)^{1-(v/b)}}{1-(v/b)} \right) + K \\
&= \frac{1}{2C} \left(\frac{(Cx)^{(v/b)+1}}{v+b} + \frac{(Cx)^{1-(v/b)}}{b-v} \right) + K,
\end{aligned}$$

where we must be careful to preserve the assumption that $Cx > 0$.

The dog starts at $x = w < 0$ when $t = 0$, so that

$$\begin{aligned}
0 &= t(w) \\
&= \frac{1}{2C} \left(\frac{(Cw)^{(v/b)+1}}{v+b} + \frac{(Cw)^{1-(v/b)}}{b-v} \right) + K,
\end{aligned}$$

and

$$K = -\frac{1}{2C} \left(\frac{(Cw)^{(v/b)+1}}{v+b} + \frac{(Cw)^{1-(v/b)}}{b-v} \right). \tag{10}$$

Note that x is always less than zero, so we must take $C < 0$, which means that $K > 0$. Also, the dog arrives at the destination when $x = 0$, so the total time required for the trip is K.

4.1 Current equals Boat Speed

The derivation above assumes that $v \neq b$. In fact, Equation (8) for the position, remains valid when $v = b$, but the equation for time as a function of x is no longer valid. In particular, K, the constant of integration, given by Equation (10), involves dividing by $b - v$.

Go back to Equation (9). It's a solution to

$$t(x) \quad = \quad \frac{1}{2bC} \int \left(w^k + \frac{1}{w^k} \right) \; dw,$$

where $k = v/b$, which is now 1. So, when $v = b$, we have

$$t(x) \quad = \quad \frac{1}{2bC} \int \left(w + \frac{1}{w} \right) \; dw,$$

and

$$t(x) \quad = \quad \frac{1}{2bC} \left(\frac{1}{2} w^2 + \ln |w| \right) + K.$$

Unwinding the substitution $w = Cx$,

$$t(x) \quad = \quad \frac{1}{2bC} \left(\frac{1}{2} (Cx)^2 + \ln(Cx) \right) + K,$$

where the absolute value can be dropped, just as above.

The constant of integration is now given by

$$\begin{aligned} 0 &= t(w) \\ &= \frac{1}{2bC} \left(\frac{1}{2} (Cw)^2 + \ln(Cw) \right) + K, \end{aligned}$$

and

$$K \quad = \quad \frac{-1}{2bC} \left(\frac{1}{2} (Cw)^2 + \ln(Cw) \right).$$

4.2 Dog Paddle with Variable Current and Boat Speed

Allow the current and boat speed to vary in strips, as usual, and apply the dog paddle strategy. Before launching into any calculation, think about what's happening. The equations above tell us where the dog is and how much time has passed if the current and boat speed are constant and the dog is always swimming toward $(0, 0)$. Over the first strip, the equations above are still accurate; the dog doesn't know that the speeds will change; he just swims toward the destination.

Once the dog reaches the second strip and the current changes, he continues to swim toward $(0, 0)$. The equations are still accurate, although the constants of integration change because the "starting point" (where the dog crosses into the new strip) and the dog's speed are now different. As the dog moves from strip to strip, we just need to recalculate the constants of integration without changing the equations involved. Put another way, we need to solve the same problem (and reach solutions of the same form) each time the dog moves from one strip to the next.

When implementing this as a computer program, it is important to note that, if there is no current, then the equations above won't work. They blow up due to divisions by zero. If there's no current, then the dog paddles in a straight line, and the solution is trivial anyway (at least for that strip).